# Reversible shell models of turbulence

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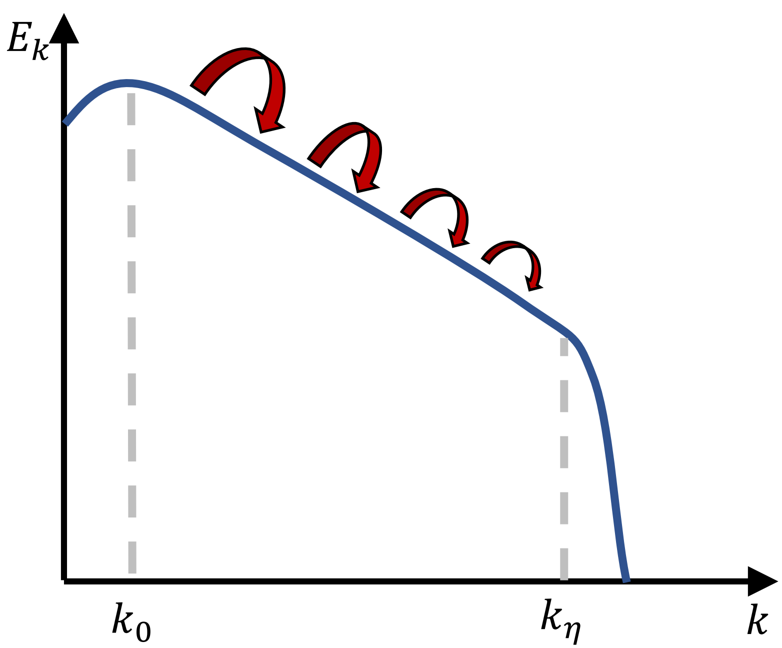
# Turbulence: phenomenology

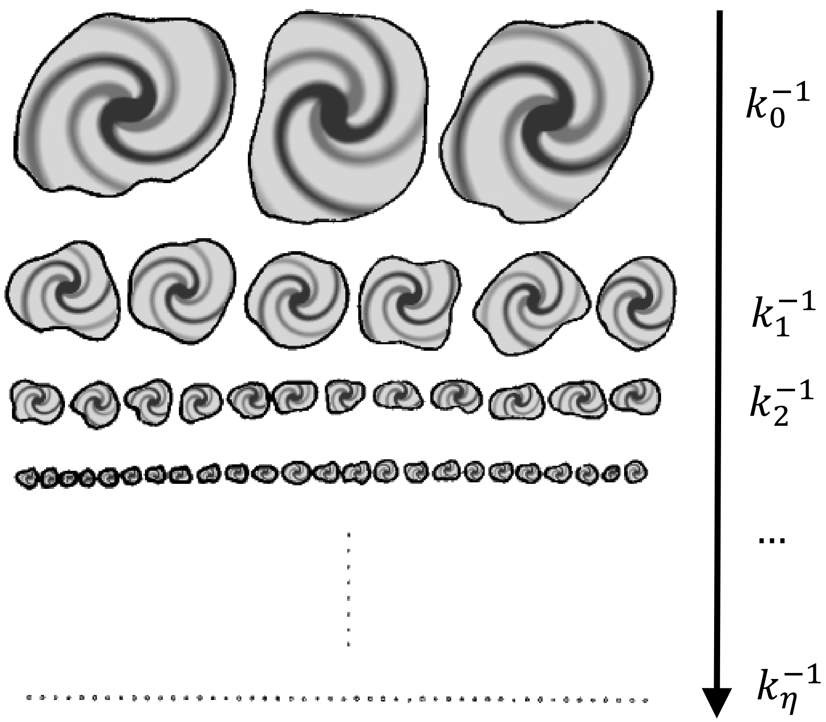
**NS equations** (3D, incompressible):

(Physical space) $\partial_t \mathbf{u}(\mathbf{x}, t) + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{F}$

(Fourier space) $\partial_t \boldsymbol{u}(\mathbf{k}, t) = -i\mathbf{k}\widehat{\Pi}(\mathbf{k}) \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \boldsymbol{u}(\mathbf{p}, t)\, \boldsymbol{u}(\mathbf{q}, t) - \nu k^2 \boldsymbol{u}(\mathbf{k}, t) + \boldsymbol{F}(\mathbf{k}, t)$

Energy cascade



# Gallavotti-Cohen equivalence conjecture

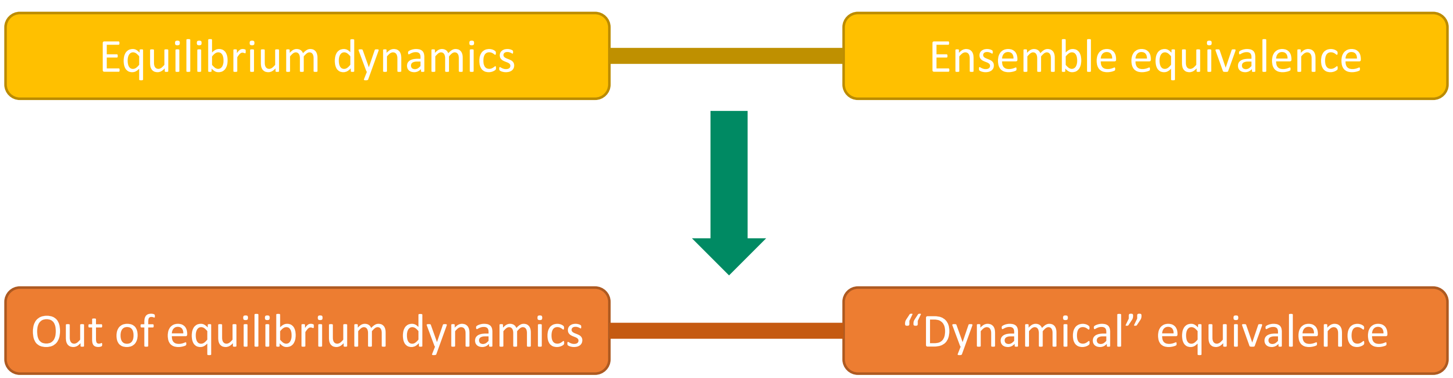
**Conjecture***

It is possible to build a time-reversible NS equation (**R-NS**), by using a configuration-dependent viscosity $\nu(\{u_{\mathbf{k}}(t)\})$ , which fixes a global quantity.

**NS** and **R-NS** are equivalent, in the sense that the **mean values of properly chosen observables are the same** in both systems, provided that:

- System is chaotic, $Re \to \infty$
- $\langle \sigma \rangle_{\mathrm{NS}} = \langle \sigma \rangle_{\mathrm{R-NS}}$

$\sigma =$ phase space contraction rate, or entropy production rate




*Gallavotti - 1997 - Dynamical ensembles equivalence in fluid mechanics. *Physica D, 105*(1)

# Shell models

Testing R-NS in direct numerical simulations requires great effort

We first try with a simpler model for turbulence: **SABRA shell model**

$$\partial_t u_n = ik_n(\lambda u_{n+2}\, u^*_{n+1} + b u_{n+1}\, u^*_{n-1} + c\, \lambda^{-1}\, u_{n-1}\, u_{n-2}) - \nu\, k_n^2\, u_n + F_n$$

**Features**:

- Discrete, logarithmically spaced shells in wavenumber space: $k_n = k_0\, \lambda^n$
- One representative velocity per shell ($u(k_n) \equiv u_n$)
- **First neighbor** interactions: $(u_{n-1}\,, u_n\,, u_{n+1})$
- **Non-linear interactions** in triads, **same as NS**
- **Physical invariants**: **Energy** and **helicity**, **conserved triad by triad, same as NS**

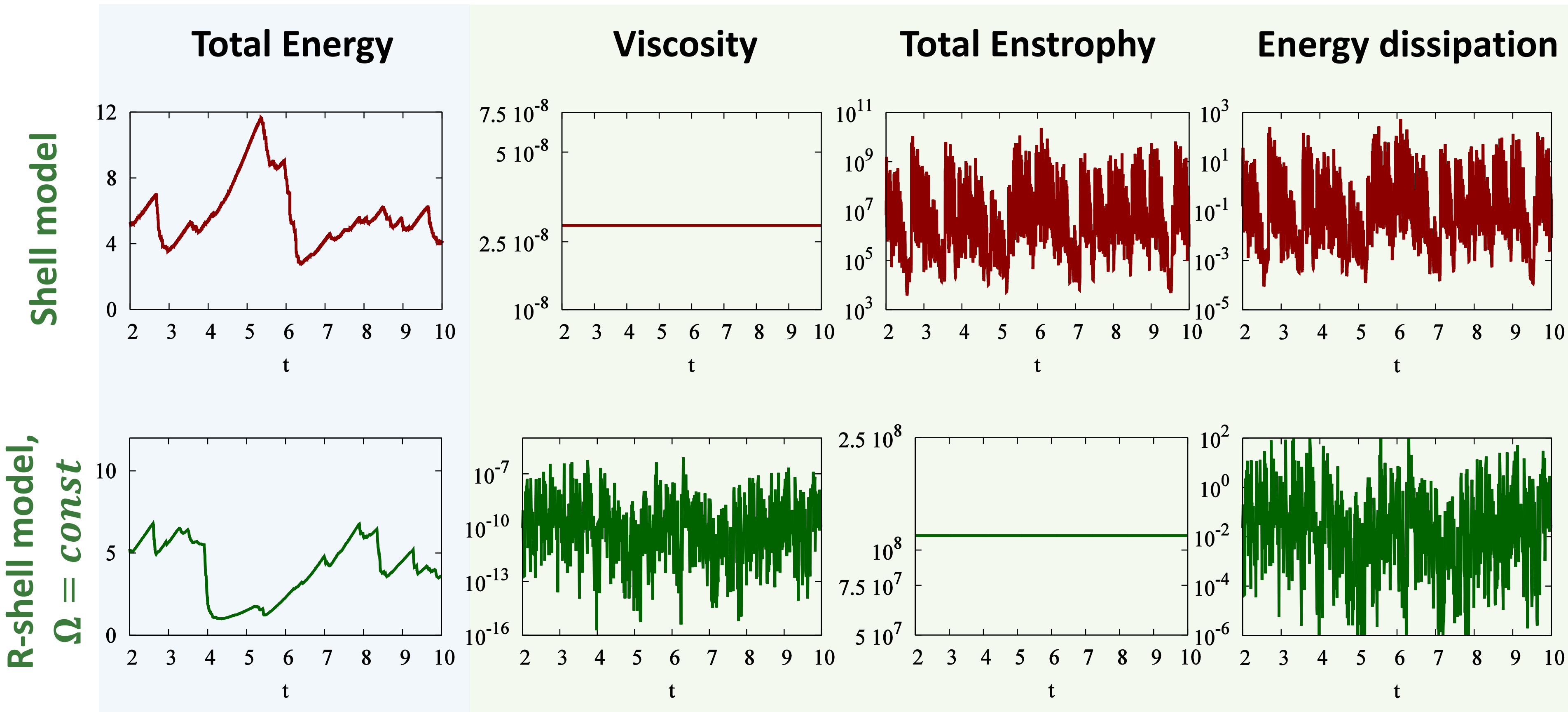
➢ Can easily reach very **high Reynolds number**

➢ Shows **energy cascade** and **anomalous scaling exponents**: $\langle |u_n|^p \rangle \sim k_n^{p/3+\delta_p}$

# R-shell model

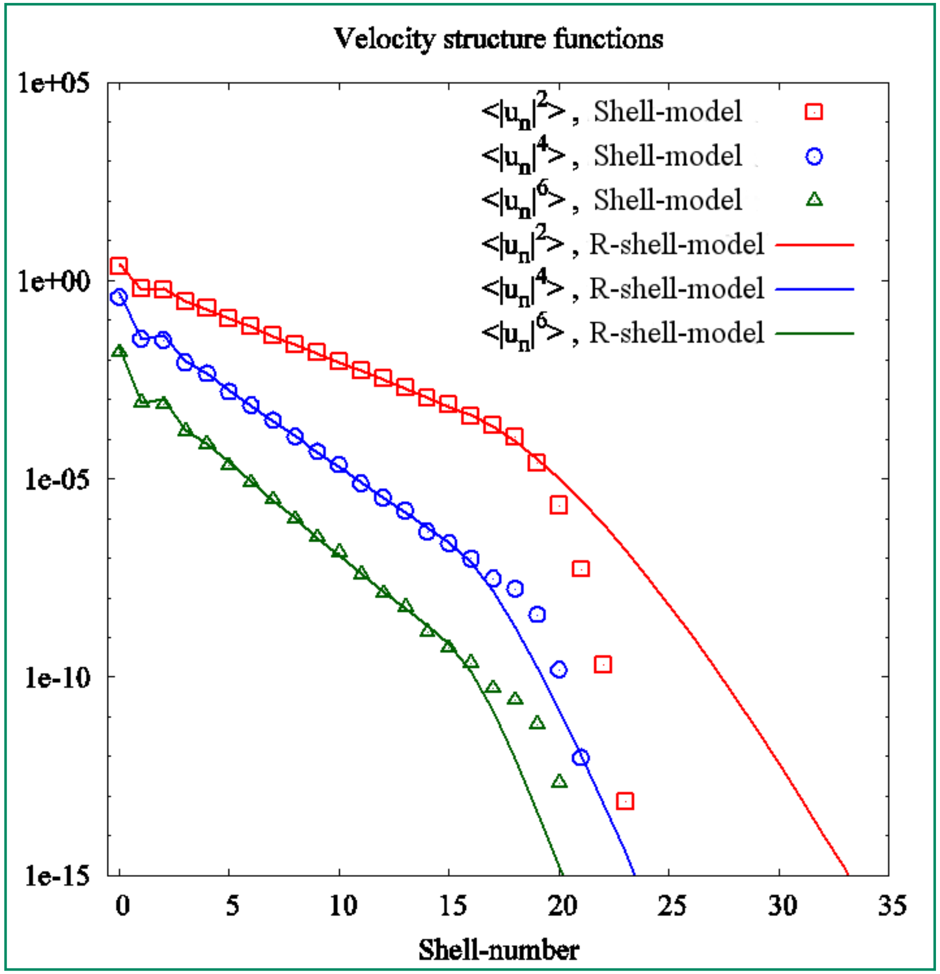
Fixed global quantity: Enstrophy $\Omega \equiv \sum_{\mathbf{k}} k^2 u_{\mathbf{k}}^2$

$$\frac{d\Omega}{dt} = 0 \quad \rightarrow \quad \nu(t) = \frac{\sum_{\mathbf{k}} k^2 Re[F_{\mathbf{k}} \cdot u_{\mathbf{k}}^*]}{\sum_{\mathbf{k}} k^4 u_{\mathbf{k}}^2} + \frac{\sum_{\mathbf{k}} k^2 Re[\mathbf{v}_{\mathbf{k}}^*(t) \cdot \boldsymbol{NLT}(\mathbf{k})]}{\sum_{\mathbf{k}} k^4 u_{\mathbf{k}}^2}$$



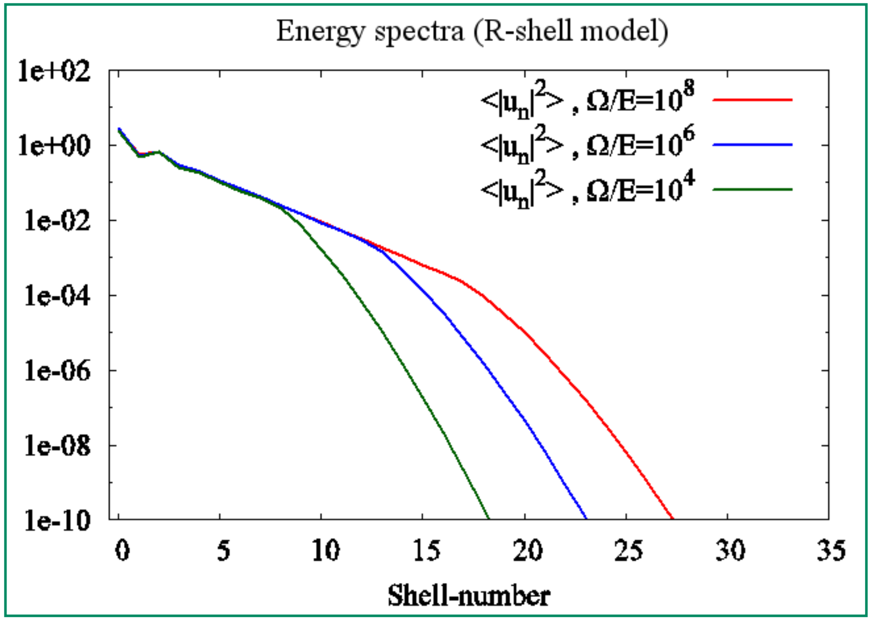
# Reversible shell model – dynamical equivalence

**The R-shell model with $\Omega = const$ works!**



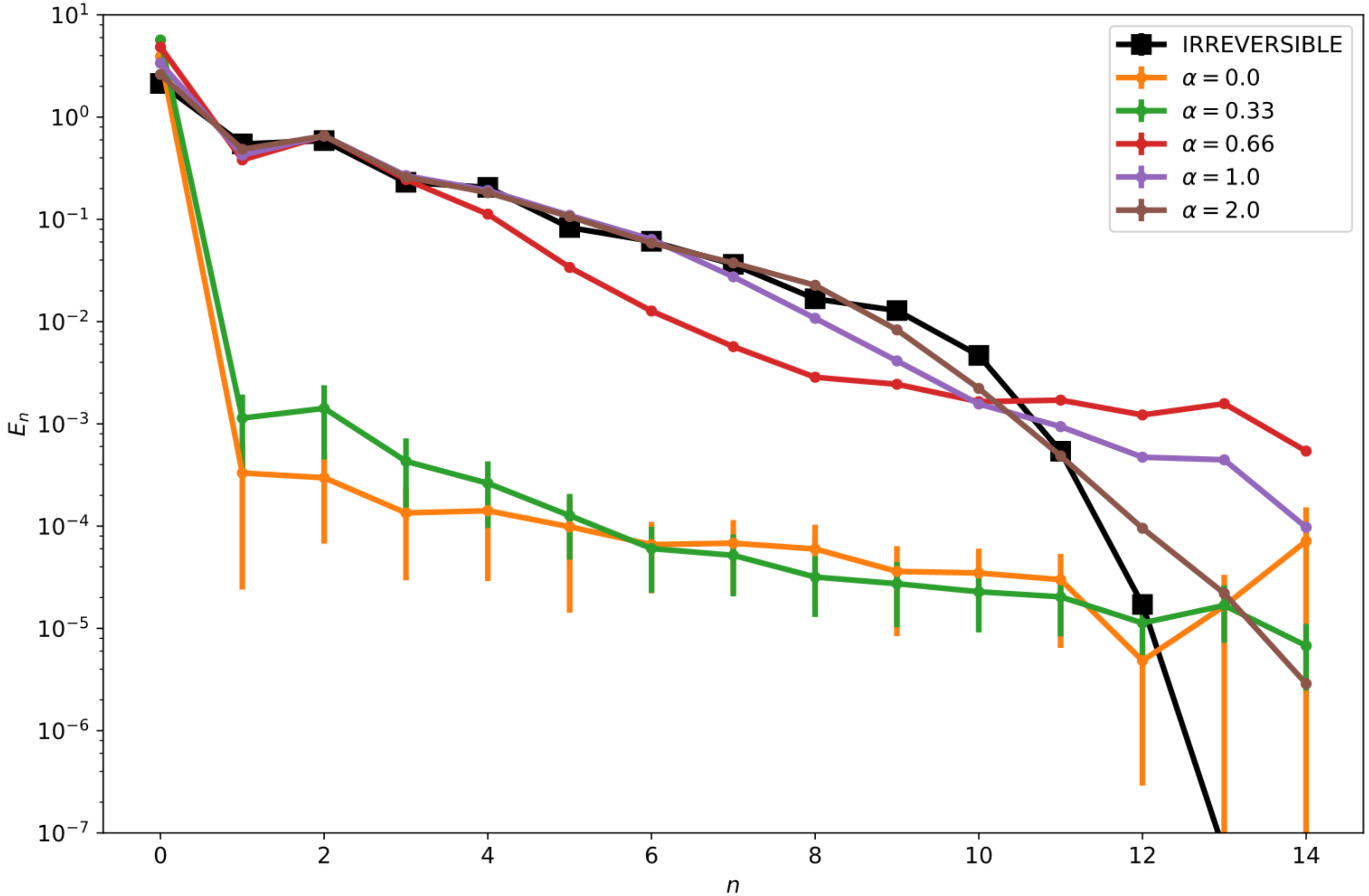

At stationary state, in the inertial range, it has:

- Kolmogorov-like energy cascade
- constant fluxes of energy and helicity
- same intermittency (anomalous exponents of the structure functions)



# Reversible shell model – dynamical equivalence

Fixed global quantity: $E^{\alpha} \equiv \sum_{\mathbf{k}} k^{\alpha} u_{\mathbf{k}}^2$




- Better equivalence (for cascade dynamics) as $\alpha$ increases

# Summary and future directions

1. Gallavotti-Cohen* equivalence for NS:
   Turbulent dynamics can be equally well represented by NS equations or R-NS, where $\nu = \nu(\{u_n(t)\})$ fixes a global quantity

2. Tested with shell models for turbulence:
   There exist range of parameters where equivalence holds
   For the energy cascade regime, better equivalence when fixed global quantity depends on small scales

Future directions:

- Robustness of the equivalence conjecture respect to spectral truncation.
  - Reversible eddy diffusivity models
- Transition from cascade to quasi-equilibrium
- In principle the same approach can be used with 3D NS

*Gallavotti - 1997 - Dynamical ensembles equivalence in fluid mechanics. *Physica D, 105*(1)